

## The spectral curve of the lens space matrix model

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## The spectral curve of the lens space matrix model

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**ABSTRACT:** Following hep-th/0211098 we study the matrix model which describes the topological A-model on  $T^*(S^3/\mathbb{Z}_p)$ . We show that the resolvent has square root branch cuts and it follows that this is a  $p$  cut single matrix model. We solve for the resolvent and find the spectral curve. We comment on how this is related to large  $N$  transitions and mirror symmetry.

**KEYWORDS:** Topological Strings, String Duality

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**1 Introduction**

The duality between open and closed string theories is a fascinating area of string theory. This duality is often understood as a geometric transition, where topologically distinct manifolds are used for the open or closed string theory, the prototypical example being the duality between the resolved and deformed conifolds [1–3]. At the level of topological string theory, for the A-model transition the closed string side is the resolved conifold and the open string side is the deformed conifold, for the B-model transition this is reversed.

The study of the B-model conifold transition and its generalizations led to the introduction of matrix models as a way to describe holomorphic Chern-Simons (HCS) theory reduced to the 2-cycles of the generalized conifold [4]. This is in turn directly related to four dimensional  $\mathcal{N} = 1$  Yang-Mills theory, because the partition function of the topological string which was used to engineer the Yang-Mills theory gives the low energy effective superpotential [5, 6]. This is now known as Dijkgraaf-Vafa (DV) theory. The connection between matrix models and superpotentials has also been uncovered directly in the field theory [7, 8].

Matrix models were introduced into the topological A-model from a very different standpoint by Marino [9], where he presented a matrix model description of Chern-Simons (CS) theory on certain 3-manifolds. This matrix model always has a quadratic potential,

but it has a rather strange measure which encodes the different geometries, when this 3-manifold is  $S^3$  this is the Haar measure on  $SU(N)$ . This work was extended in [10] where they considered the A-model open topological string on  $T^*(S^3/\mathbb{Z}_p)$  (corresponding to CS theory on  $S^3/\mathbb{Z}_p$  [11]) and also the mirror geometry ( $\tilde{X}$ ). By using similar reasoning as in [4] they were able to derive a matrix model for HCS theory reduced to  $\mathbb{P}^1$ 's in  $\tilde{X}$ . As expected but still quite remarkably, for each  $p$  HCS on  $\tilde{X}$  and CS theory on  $S^3/\mathbb{Z}_p$  are described by identical matrix models. Many of the ideas at work here (pre matrix model) are covered in the great review paper by Marino [12].

So essentially, by studying the topological A-model and using mirror symmetry, Dijkgraaf-Vafa (DV) theory was extended to a new class of Calabi-Yau manifolds. Now by the general principles of DV theory, special geometry on the closed string dual geometry of  $\tilde{X}$  (call it  $X$ ), should reduce to special geometry on a Riemann surface in  $X$ , and this surface should be the spectral curve of the aforementioned matrix model. For the case of the A-model on  $T^*S^3$ , the spectral curve was shown to coincide with the non-trivial Riemann surface in  $X$  and the leading order (in  $g_s$ ) free energy of the matrix model was shown to agree with the known result. In [13] the free energy of this matrix model was calculated to all orders and shown to agree with known results [1]. In [14] the orientifold of the conifold was considered and the subleading order free energy was shown to agree with known results [15].

In this paper we investigate the matrix model of CS theory on  $S^3/\mathbb{Z}_p$ . It was shown in [10] that this matrix model has  $p$  cuts each at the position of a  $\mathbb{P}^1$  in the blown up 3-fold and it was noticed that this model looks similar to a  $p$ -matrix model. We will show that the resolvent has square root cuts which implies that really it is a single matrix model with  $p$  cuts. We then find that the spectral curve is a genus  $(p - 1)$  Riemann surface with four points deleted and find the equation for this curve. For the case of  $p = 2$  we compare our surface to that obtained from the Hori-Vafa mirror.

This paper is organized as follows. In section 2. we discuss the geometrical structures which are involved in the large  $N$  duality we are considering and in mirror symmetry of these dualities. In section 3 we review the solution of the matrix model for CS theory on  $S^3$  and also solve it with our new method. In sections 4 and 5 we solve the case of  $S^3/\mathbb{Z}_2$  and  $S^3/\mathbb{Z}_p$  respectively. In section 6. we outline our calculation of the free energy for  $S^3/\mathbb{Z}_2$  which we view as a non-trivial check of our method, the full calculation is presented in the appendix.

## 2 Geometry

The first geometric transition to be studied was the A-model conifold transition of Gopakumar and Vafa [1]. They considered the closed topological A-model on the resolved conifold  $(\mathcal{O}_{-1} + \mathcal{O}_{-1} \rightarrow \mathbb{P}^1)$  and argued that it is equivalent to the open topological A-model on the deformed conifold  $(T^*(S^3))$ . This has been extended significantly to a large class of toric Calabi-Yau's [16–18] but not including the expected transition between  $A_{p-1} \rightarrow \mathbb{P}^1$  and  $S^3/\mathbb{Z}_p$ .

Heuristically, taking a  $\mathbb{Z}_p$  orbifold on both sides of the A-model conifold transition should produce a transition between some  $A_{p-1}$  fibration over  $\mathbb{P}^1$  on the closed string side

and  $T^*(S^3/\mathbb{Z}_p)$  on the open string side. In [10] the matrix model of CS on  $S^3/\mathbb{Z}_2$  was studied, its free energy was calculated perturbatively and was shown to agree with the closed A-model on  $\mathcal{O}(-K) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  (which is a trivial  $A_1$  fibration over  $\mathbb{P}^1$ ), thus implying that  $T^*(S^3/\mathbb{Z}_2)$  undergoes a geometric transition to  $\mathcal{O}(-K) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . For the case of  $T^*(S^3/\mathbb{Z}_p)$  the Riemann surface embedded in  $X$  is given by the Hori-Vafa mirror map, which we will now briefly describe.

By studying T-duality on  $T^3$  fibres of an arbitrary toric Calabi-Yau manifold [19], a mirror map was derived. This map can be reduced to the following operation. Take a toric web diagram<sup>1</sup> of a toric CY threefold  $M$ , then consider the Riemann surface obtained by thickening each line into a cylinder. This Riemann surface will be

$$F(e^u, e^v) = 0, \tag{2.1}$$

and then the 3-fold mirror to  $M$  is given by

$$xy = F(e^u, e^v). \tag{2.2}$$

The Hori-Vafa mirror map gives  $F(e^u, e^v)$  explicitly. The various geometries and dualities involved here are shown in figure 1.

This map gives the mirror to the resolved conifold  $(\mathcal{O}_{-1} + \mathcal{O}_{-1} \rightarrow \mathbb{P}^1)$  to be

$$xy = (e^v - 1)(e^{u+v} - 1) - 1 + e^t \equiv F_c(e^u, e^v) \tag{2.3}$$

and the spectral curve of the appropriate matrix model (CS theory on  $S^3$ ) was shown in [10] to be given by  $F_c(e^u, e^v) = 0$ . In this paper we further study this by reconsidering the general case of CS theory on  $T^*(S^3/\mathbb{Z}_p)$ . The mirror  $(\tilde{X})$  of  $T^*(S^3/\mathbb{Z}_p)$  is given by blowing up the singular 3-fold

$$xy = (e^v - 1)(e^{v+pu} - 1) \equiv F_p(e^u, e^v). \tag{2.4}$$

We will find that the associated matrix model has a spectral curve which is a genus  $(p-1)$  Riemann surface with four points deleted, given by a certain complex structure deformation of  $F_p(e^u, e^v) = 0$ . Whilst we cannot calculate precisely the complex structure parameters, for the case  $p=2$  we can give an expansion in the 't Hooft parameters, which in principle could be generalized to  $p > 2$ . We also find which monomials appear in the deformation. All this relies on our showing that the matrix model is a single matrix model with  $p$  square root cuts.<sup>2</sup>

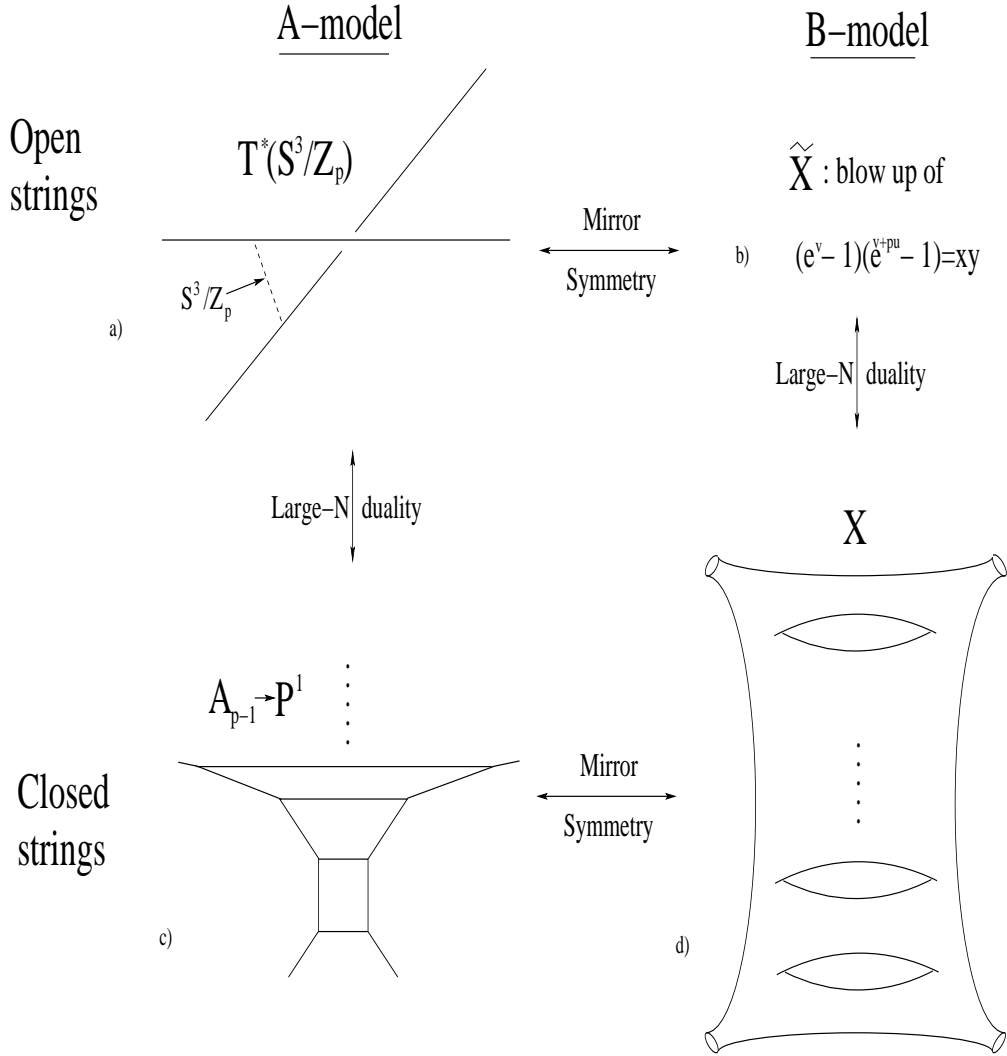
### 3 Chern-Simons matrix model on $S^3$

In this section we review the matrix model that describes CS theory on  $S^3$  [10]. We study this model with a new method we have developed, a method which generalizes nicely to  $S^3/\mathbb{Z}_p$

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<sup>1</sup>a toric web diagram is a trivalent graph such that three unit vectors emanating from each node sum to zero [20], it encodes the singular structure of the  $T^3$  fibration.

<sup>2</sup>In [10] it was suggested that one could view it as a  $p$ -matrix model, this would produce a  $p$ -sheeted Riemann surface as for the quiver matrix model [21–23].



**Figure 1.** Large N dualities and mirror symmetry. a)  $T^*(S^3/\mathbb{Z}_p)$  represented by a deformation of a toric diagram [25] b) The mirror to  $T^*(S^3/\mathbb{Z}_p)$  c) Schematic picture of the toric web of an  $A_{p-1}$  fibration over  $\mathbb{P}^1$  d) The Hori-Vafa mirror map gives a bundle over a genus  $p - 1$  Riemann surface, where the Riemann surface is simply a thickening of the toric web diagram

### 3.1 Solution by contour integral

The matrix integral is given by

$$Z \sim \int \prod_i du_i \Delta^2(u) \exp\left(-\frac{1}{g_s} \sum_i u_i^2/2\right) \tag{3.1}$$

where the group measure is an analytic continuation of the Haar measure,

$$\Delta(u) = \prod_{i < j} 2 \sinh\left(\frac{u_i - u_j}{2}\right). \tag{3.2}$$

Although the measure is periodic, the potential is not, therefore the domain of integration is non-compact. The equation of motion for each eigenvalue is

$$\frac{1}{g_s} u_i = \sum_{j \neq i} \coth \left( \frac{u_i - u_j}{2} \right). \quad (3.3)$$

In general, the form of the resolvent can be inferred from the function on the r.h.s. of the equations of motion and the measure (3.2). In this case it is

$$\omega(z) = g_s \sum_i \coth \left( \frac{z - u_i}{2} \right). \quad (3.4)$$

We then multiply (3.3) by  $\coth((z - u_i)/2)$ , sum over eigenvalues and take the large N limit. This leads to the following loop equation

$$\left( \frac{\omega(z)}{2} \right)^2 - z \frac{\omega(z)}{2} = f(z) + \frac{1}{4} S^2, \quad (3.5)$$

with

$$f(z) = \frac{1}{2} g_s \sum_i (u_i - z) \coth \left( \frac{z - u_i}{2} \right) \quad (3.6)$$

being a regular function. Eq. (3.5) shows that the resolvent acquires a square root cut in the large N limit and that there is only one cut.

The spectral curve is obtained by gluing two infinite cylinders along the cut. There are two independent cycles. The  $A$  cycle is a contour around the cut, the  $B$  cycle starts at infinity on the classical sheet where the resolvent is finite and goes to the other sheet through the cut. We call  $S = g_s N$  the 't Hooft parameter. From equation (3.4), we get the limiting value of the resolvent,

$$\lim_{z \rightarrow \infty} \omega(z) = S \quad (3.7)$$

One also has to fix the period over the  $A$ -cycle

$$\pi i S = \oint_A \frac{\omega(z)}{4} dz. \quad (3.8)$$

The last condition is equivalent to  $\tilde{\omega}(z)$ , the other branch of the resolvent, having the limiting value

$$\lim_{z \rightarrow -\infty} \tilde{\omega}(z) = -S. \quad (3.9)$$

This agrees with the definition (3.4). All these conditions are sufficient to find the resolvent. We review briefly how it is done in [10].

The equation of motion can be written a little differently by introducing a new resolvent

$$v(Z) \equiv g_s \sum_i \frac{U_i}{U_i - Z} \quad (3.10)$$

with  $U_i = e^{u_i}$  and  $Z = e^z$ . Importantly, both  $v(Z)$  and  $\omega(Z)$  have the same singular behaviour, the relation between them is given by

$$\omega(Z) = S - 2v(Z). \tag{3.11}$$

The problem is now essentially a Hermitian matrix model with a logarithmic potential, which leads by standard arguments [24] to

$$-2v(Z) = \sqrt{(Z-a)(Z-b)} \oint_C \frac{dX \log(Xe^{-S})}{2\pi i} \frac{1}{X-Z} \frac{1}{\sqrt{(X-a)(X-b)}}, \tag{3.12}$$

where the contour  $C$  encircles the cut but not the point  $Z$ . The normalization conditions at  $\pm\infty$  fix the end points of the cut and the final answer is

$$\omega(Z) = \log \left( \frac{e^{-S/2}}{2} \left( Z + 1 - \sqrt{(1+Z)^2 - 4Ze^S} \right) \right). \tag{3.13}$$

The spectral curve is the surface where the resolvent is well defined, in this case it is given by (as advertised after eq. (2.3))

$$(e^v - 1)(e^{u+v} - 1) + e^S - 1 = 0, \tag{3.14}$$

where,  $u \equiv z$ .

Although this procedure is not very lengthy it becomes difficult even for the case of  $S^3/\mathbb{Z}_2$  lens space.

### 3.2 Solution by a regular function

We have found another way of finding the resolvent, this method will easily generalize to the case of  $S^3/\mathbb{Z}_p$ .

Let  $\omega_+$  be the value of the resolvent on one edge of the cut and  $\omega_-$  be the value of the resolvent on the other edge. From the large  $N$  limit of (3.3), it is clear that

$$\frac{\omega_+(z)}{2} + \frac{\omega_-(z)}{2} = z. \tag{3.15}$$

We then construct the function

$$g(Z) \equiv e^{\omega/2} + Ze^{-\omega/2}. \tag{3.16}$$

which is regular everywhere except at infinity. The limiting behavior of the resolvent will completely determine this function,

$$\lim_{Z \rightarrow \infty} g(Z) = e^{-S/2} Z, \tag{3.17}$$

$$\lim_{Z \rightarrow 0} g(Z) = e^{-S/2}. \tag{3.18}$$

The unique function that satisfies these conditions is

$$g(Z) = e^{-S/2}(Z + 1). \tag{3.19}$$



Now the quadratic equation (3.16) gives the resolvent explicitly,

$$e^{\omega/2} = \frac{1}{2} \left( g(Z) - \sqrt{g^2(Z) - 4Z} \right). \tag{3.20}$$

This is the same resolvent as (3.13) that is obtained using the contour integral representation and thus we get the same spectral curve (3.14).

Let us summarize the main strategy. From the large  $N$  limit of the equation of motion we can deduce that the resolvent has a square root cut, then the value of the resolvent on one edge of the cut can be simply related to the value of the resolvent on the other edge of the cut. Knowing this, one has to construct a function of the resolvent that is regular everywhere except infinity. The main ingredients are the functions  $e^{\omega/2}$  and  $e^{-\omega/2}$ . Once such a function is found,  $e^{\omega/2}$  can be written as a solution to a quadratic equation. This strategy will be shown to work for all Lens spaces. For  $S^3/\mathbb{Z}_p$  with  $p$  even, it is also possible to construct a function which is square root branched on each cut and from this, solve for  $\omega(Z)$ .

#### 4 $S^3/\mathbb{Z}_2$ Lens space resolvent

We now employ the strategy from the previous section for the geometry  $S^3/\mathbb{Z}_2$ . We refer the reader to [10] for a derivation of the Lens space matrix model but the reader can also just take (4.1) as a starting point. The partition function for CS theory on  $S^3/\mathbb{Z}_2$  is given by the integral over two sets of eigenvalues

$$Z \sim \int \prod_i du_i \prod_\alpha d\mu_\alpha \Delta^2(u, \mu) \exp \left( -\frac{1}{g_s} V(u, \mu) \right), \tag{4.1}$$

where the measure is

$$\Delta(u, \mu) = \prod_{i < j} 2 \sinh \left( \frac{u_i - u_j}{2} \right) \prod_{\alpha < \beta} 2 \sinh \left( \frac{\mu_\alpha - \mu_\beta}{2} \right) \prod_{i, \alpha} 2 \cosh \left( \frac{u_i - \mu_\alpha}{2} \right) \tag{4.2}$$

and  $i \in (1, N_1)$ ,  $\alpha \in (1, N_2)$ . Anticipating taking the large  $N$  limit we also introduce two 't Hooft parameters  $S_1 = g_s N_1$  and  $S_2 = g_s N_2$  and  $S = S_1 + S_2$ . The potential is

$$V(u, \mu) = \left( 2 \sum_i u_i^2 + 2 \sum_\alpha \mu_\alpha^2 \right) / 2, \tag{4.3}$$

and the equations of motion for each eigenvalue are

$$2u_i = g_s \sum_{j \neq i} \coth \left( \frac{u_i - u_j}{2} \right) + g_s \sum_\alpha \tanh \left( \frac{u_i - \mu_\alpha}{2} \right) \tag{4.4}$$

$$2\mu_\alpha = g_s \sum_{\beta \neq \alpha} \coth \left( \frac{\mu_\alpha - \mu_\beta}{2} \right) + g_s \sum_i \tanh \left( \frac{\mu_\alpha - u_i}{2} \right). \tag{4.5}$$

We define the resolvents as

$$\omega(z) = g_s \sum_i \coth\left(\frac{z - u_i}{2}\right) + g_s \sum_\alpha \tanh\left(\frac{z - \mu_\alpha}{2}\right). \quad (4.6)$$

$$\omega_1(z) = g_s \sum_i \coth\left(\frac{z - u_i}{2}\right), \quad (4.7)$$

$$\omega_2(z) = g_s \sum_\alpha \coth\left(\frac{z - \mu_\alpha}{2}\right), \quad (4.8)$$

so the relation between them reads

$$\omega(z) = \omega_1(z) + \omega_2(z - i\pi). \quad (4.9)$$

#### 4.1 Solution by a regular function

Now we multiply equation (4.4) by ‘ $\coth((z - u_i)/2)$ ’ and sum over  $i$ , as well as multiplying equation (4.5) by ‘ $\tanh((z - \mu_\alpha)/2)$ ’ and summing over  $\alpha$ . Then we add these two equations and take the large  $N$  limit, with the result being

$$\left(\frac{\omega(z)}{2}\right)^2 - 2z\frac{\omega_1(z)}{2} - 2(z - i\pi)\frac{\omega_2(z - i\pi)}{2} = f(z) \quad (4.10)$$

where

$$f(z) = g_s \sum_i (u_i - z) \coth\left(\frac{z - u_i}{2}\right) + g_s \sum_\alpha (\mu_\alpha - (z - i\pi)) \tanh\left(\frac{z - \mu_\alpha}{2}\right) + \frac{1}{4}S^2 \quad (4.11)$$

is a regular function. We can write (4.10) in two ways,

$$\left(\frac{\omega(z)}{4}\right)^2 - (z - i\pi)\frac{\omega(z)}{4} - i\pi\frac{\omega_1(z)}{4} = \frac{f(z)}{4}, \quad (4.12)$$

$$\left(\frac{\omega(z + i\pi)}{4}\right)^2 - (z + i\pi)\frac{\omega(z + i\pi)}{4} + i\pi\frac{\omega_2(z)}{4} = \frac{f(z + i\pi)}{4}. \quad (4.13)$$

Now we make an important assumption, we assume that the eigenvalues spread only along the real line. For general multi matrix models this is not true [21–23]. However as we will see this assumption leads to the correct result for our case. It follows that if  $\omega_1(z)$  jumps at a point  $z$  then  $\omega_2(z - i\pi)$  does not and vice versa. Note that we do not make any assumption on the type of the cuts. In the total resolvent  $\omega(z)$ , the individual resolvents come with a relative shift of the argument by  $i\pi$ . Therefore the two cuts in the total resolvent are now separated by  $i\pi$ . On one cut the total resolvent jumps only due to  $\omega_1(z)$  and on the other cut only due to  $\omega_2(z)$ . From this we can deduce that

$$\frac{1}{4}(\omega_+(z) + \omega_-(z)) = z \quad (\text{u cut}) \quad (4.14)$$

$$\frac{1}{4}(\omega_+(z + i\pi) + \omega_-(z + i\pi)) = z \quad (\mu \text{ cut}) \quad (4.15)$$

and so the resolvent  $\omega(z)$  really does have square root branch cuts.<sup>3</sup> Using (4.14), (4.15),<sup>4</sup> it is straightforward to find a function of  $\omega(z)$  which is regular everywhere except at infinity, it is

$$g(Z) \equiv e^{\omega/2} + Z^2 e^{-\omega/2}. \tag{4.16}$$

This function is regular and has limiting behavior

$$\lim_{Z \rightarrow \infty} g(Z) = Z^2 e^{-S/2}, \tag{4.17}$$

$$\lim_{Z \rightarrow 0} g(Z) = e^{-S/2}. \tag{4.18}$$

Therefore  $g(Z)$  can be written in terms of only one unknown parameter

$$g(Z) = e^{-S/2}(Z^2 + dZ + 1), \tag{4.19}$$

where  $d$  is related to the end points of the cuts.

Solving (4.16) as a quadratic equation for  $e^{\omega/2}$  yields,

$$\frac{\omega(Z)}{2} = \log \left( \frac{1}{2} \left( g(Z) - \sqrt{g^2(Z) - 4Z^2} \right) \right). \tag{4.20}$$

It is easy to see that  $\frac{1}{2}(\omega_+(Z) + \omega_-(Z)) = \log(Z^2)$  and therefore (4.14) and (4.15) are satisfied.

Now consider the function under the square root sign in (4.20). If  $Z_i$  sets this to zero, then  $1/Z_i$  will as well. Together with fact that the eigenvalues are all real, this implies that the end points of each cut are the inverse of one another, i.e. the  $u$  cut is  $(a, 1/a)$ , the  $\mu$  cut is  $(b, 1/b)$  for some  $a, b$ . Further, the relationship between our parameter  $d$  and the end point of the cuts is easy to find

$$d = 2e^{S/2} - \left( a + \frac{1}{a} \right), \tag{4.21}$$

$$d = -2e^{S/2} + \left( b + \frac{1}{b} \right). \tag{4.22}$$

As discussed in the previous section, the spectral curve is two cylindrical sheets glued together along these cuts. The center of the  $u$  cut is at  $z = 0$  and the center of the  $\mu$  cut is located at  $z = i\pi$ .

Let's call the contour around the  $u$  cut the  $A_1$  cycle and the contour around the  $\mu$  cut the  $A_2$  cycle. There are also two dual  $B$  cycles. The  $B_1$  cycle starts at a point  $\Lambda$  at infinity on the classical sheet where the resolvent is finite and goes to a point  $\tilde{\Lambda}$  on the second sheet through the  $u$  cut. The end points of the  $B_2$  cycle are the same but the contour goes from one sheet to the other through the  $\mu$  cut. The Riemann surface is depicted on the figure 2.

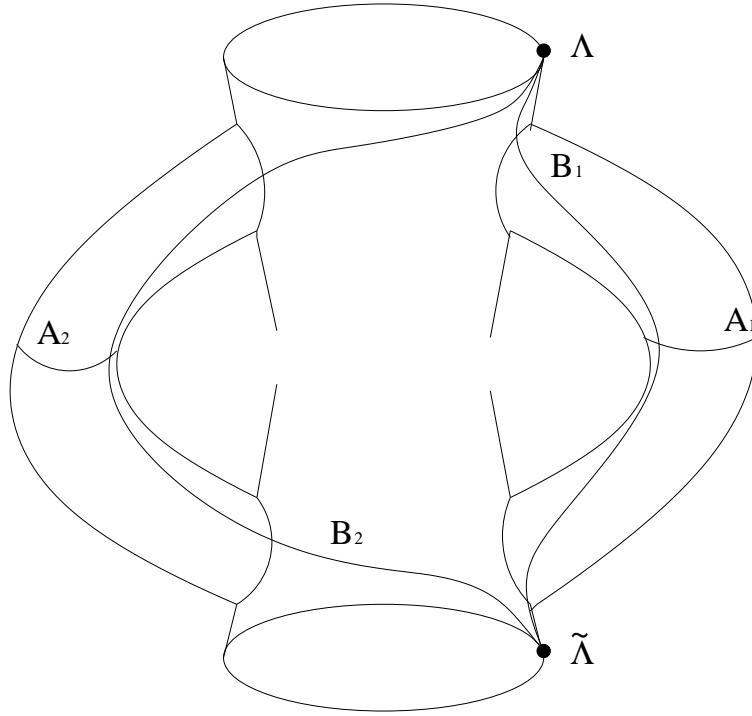
Now to find  $a$ , the end point of the  $u$  cut, one has to fix the period over the  $A_1$  cycle

$$\frac{1}{4} \oint_{A_1} \omega(z) dz = \pi i S_1. \tag{4.23}$$

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<sup>3</sup>due to the fact that the matrix model looks much like a 2-matrix model, the concern was that it may be branched by a cubic root.

<sup>4</sup>which should be thought of as a principle value integral.



**Figure 2.** Spectral curve for  $S^3/\mathbb{Z}_2$  matrix model.

Analogously, the period over the  $A_2$  cycle must be proportional to  $S_2$

$$\frac{1}{4} \oint_{A_2} \omega(z) dz = \pi i S_2. \tag{4.24}$$

Actually given the normalization condition at  $z = -\infty$ , only one of those periods is independent. The integral over the  $A = A_1 + A_2$  cycle is fixed by

$$\frac{1}{4} \oint_A \omega(z) dz = -\pi i \tilde{\omega}(-\infty) = \pi i S, \tag{4.25}$$

therefore to fix  $a$  we have exactly one integral to do, either the  $A_1$  period or  $A_2$  period. These period integrals are hard to take in an explicit form, we will use a perturbative method to calculate them.

The deformed CY is given explicitly from (4.20) with  $u \equiv z$  and  $v = (S - \omega)/2$  as

$$(e^v - 1)(e^{2u+v} - 1) + e^S - 1 - de^{u+v} = xy \tag{4.26}$$

which is a particular complex structure deformation of  $F_2 = xy$  (from eq. (2.4)). The mirror of  $\mathcal{O}(-K) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is given by [26]

$$xy = e^u + e^v + e^{-t-u} + e^{-s-u} + 1 \tag{4.27}$$

where  $t$  and  $s$  are complex structure moduli. There is a simple coordinate transformation that brings (4.26) to (4.27). Explicitly  $v \rightarrow v + \ln d - S + u + i\pi$  and  $u \rightarrow u - \ln d$  which gives

us the following relationship between complex structure moduli and 't Hooft parameters

$$t = \ln d(S_1, S_2) \quad (4.28)$$

$$s = 2 \ln d(S_1, S_2) - S. \quad (4.29)$$

It concludes that the matrix model spectral curve is indeed what we expect from the mirror symmetry. In the section 6 we find perturbative expression for the complex structure deformation parameter  $d(S_1, S_2)$  and for the free energy using the resolvent (4.20). Perturbative calculations are valid when the values of 't Hooft parameters are small. Notice that the coordinate transformation above is not regular when 't Hooft parameters goes to 0 since  $d$  is also small in this limit. Therefore one can not use a perturbative expression for  $d$  to relate it to the Kahler parameters  $Re(t)$  and  $Re(s)$  using (4.28) and (4.29).

## 5 General $S^3/\mathbb{Z}_p$ lens spaces

We now generalize this analysis to the case  $S^3/\mathbb{Z}_p$ . Here there are  $p$  sets of eigenvalues, we label them by an index  $I \in \{0, \dots, p-1\}$ . The measure factor is a product of two factors, a self interacting term ( $\Delta_1$ ) and a term containing the interaction between different sets of eigenvalues ( $\Delta_2$ ),

$$\Delta_1(u) = \prod_I \prod_{i \neq j} \left( 2 \sinh \left( \frac{u_i^I - u_j^I}{2} \right) \right)^2 \quad (5.1)$$

$$\Delta_2(u) = \prod_{I < J} \prod_{i,j} \left( 2 \sinh \left( \frac{u_i^I - u_j^J + d^{IJ}}{2} \right) \right)^2, \quad (5.2)$$

where  $d^{IJ} = 2\pi i(I - J)/p$ . The potential has an overall factor of  $p$  compared to the  $S^3$  case,

$$V(u) = p \sum_{I,i} \frac{(u_i^I)^2}{2}. \quad (5.3)$$

We define individual resolvents for each set of the eigenvalues by

$$\omega_I(z) = g_s \sum_i \coth \left( \frac{z - u_i^I}{2} \right) \quad (5.4)$$

and the total resolvent, which we are most interested in is

$$\omega(z) = \sum_I \omega_I \left( z - \frac{2\pi i I}{p} \right). \quad (5.5)$$

The equation of motion for each eigenvalue is

$$p u_i^I = g_s \sum_{i \neq j} \coth \left( \frac{u_i^I - u_j^I}{2} \right) + g_s \sum_{J \neq I} \sum_j \coth \left( \frac{u_i^I - u_j^J + d^{IJ}}{2} \right). \quad (5.6)$$

From the large  $N$  limit of this equation we can derive

$$\frac{1}{2}\omega^2(z) - p \sum_I \left( z - \frac{2\pi i I}{p} \right) \omega_I \left( z - \frac{2\pi i I}{p} \right) = f(z), \tag{5.7}$$

where  $f(z)$  is a regular function. From this it follows that

$$\frac{1}{2} \left( \omega_+ \left( z + \frac{2\pi i I}{p} \right) + \omega_- \left( z + \frac{2\pi i I}{p} \right) \right) = pz, \quad (I\text{'th cut}). \tag{5.8}$$

and so every cut is indeed a square root. Now we construct a regular function,

$$g(Z) = e^{\omega/2} + Z^p e^{-\omega/2}, \tag{5.9}$$

which has the limiting behavior,

$$\lim_{Z \rightarrow \infty} g(Z) = e^{-S/2} Z^p \tag{5.10}$$

$$\lim_{Z \rightarrow 0} g(Z) = e^{-S/2} \tag{5.11}$$

and is thus of the form,

$$g(Z) = e^{-S/2} (Z^p + d_{p-1} Z^{p-1} + \dots + d_1 Z + 1). \tag{5.12}$$

The function  $g(Z)$  depends on  $p-1$  moduli  $d_n$ , which could be found by evaluating the period integrals

$$\frac{1}{2} \oint_{A_I} \omega(z) dz = 2\pi i S_I. \tag{5.13}$$

Since we have already fixed the integral over the cycle  $A = \sum_I A_I$ , there are only  $p-1$  independent  $A$ -periods.

We can solve (5.9) for  $\omega(Z)$  to get

$$\frac{\omega(Z)}{2} = \log \left( \frac{1}{2} \left( g(Z) - \sqrt{g^2(Z) - 4Z^p} \right) \right), \tag{5.14}$$

the function under the square root is a polynomial of the degree  $2p$ , it has  $2p$  distinct roots that depend on only  $p-1$  parameters. Thus the spectral curve consists of two cylinders glued together along  $p$  cuts. Note that the center of the  $I$ 'th cut is at the point  $z = 2\pi i I/p$ . From (5.14) we see that the spectral curve is given by

$$(e^v - 1)(e^{pu+v} - 1) + e^S - 1 + e^v \sum_{n=1}^{p-1} d_n e^{nu} = 0, \tag{5.15}$$

a complex structure deformation of  $F_p = 0$  (from (2.4)).

## 6 Free energy

An important check of our calculations is to use the resolvent we have found to calculate the free energy perturbatively. In the appendix we perform this for  $p = 2$ , here we quote our result, it agrees with that obtained in [10].

From (4.19)–(4.22) we can see that

$$\frac{\omega(z)}{4} = \log \left( \frac{e^{-S/4}}{2} [\sqrt{(Z+b)(Z+1/b)} - \sqrt{(Z-a)(Z-1/a)}] \right). \quad (6.1)$$

It is not possible to obtain the parameter  $a$  as an explicit function of the 't Hooft parameters but we can find a perturbative series for it. We will do this by introducing two small parameters  $\epsilon_1$  and  $\epsilon_2$  in the following way

$$a + \frac{1}{a} = 2(1 + \epsilon_1), \quad b + \frac{1}{b} = 2(1 + \epsilon_2) \quad (6.2)$$

and then performing the  $A$  period integrals as an expansion in  $\epsilon_1$  and  $\epsilon_2$ . This will give the 't Hooft parameters as a power series in  $\epsilon_1, \epsilon_2$  which we then invert. We find that

$$\begin{aligned} \epsilon_1 = & S_1 + \frac{1}{4}S_1(S_1 + S_2) + \frac{1}{96}S_1(3S_2^2 + 9S_1S_2 + 4S_1^2) + \\ & + \frac{1}{384}S_1(S_2^3 + 6S_2^2S_1 + 7S_2S_1^2 + 2S_1^3) \end{aligned}$$

and so we see that when  $S_1 = 0$ ,  $\epsilon_1 = 0$  and so the second square root in (6.1) becomes a complete square thus there is only one cut. This agrees with the fact that if the second cut is empty the problem should reduce to CS theory on  $S^3$ . The corresponding expression for  $\epsilon_2$  can be obtained from (B.15) by switching  $S_1$  and  $S_2$ .

By performing this expansion of the resolvent in  $\epsilon_1$  and  $\epsilon_2$  and then calculating the  $B$  period integrals, we can get an expansion for the free energy. This analysis is also done in the appendix, we quote the result

$$\partial_{S_1} F_0(S_1, S_2) = -S_1(1 + \log 2) + 2S_2 \log 2 + S_1 \log S_1 + \frac{1}{8}(S_1 + S_2)^2 + \quad (6.3)$$

$$+ \frac{1}{576}(3S_2^3 + 18S_2^2S_1 + 9S_2S_1^2 + 2S_1^3) + \mathcal{O}(S^5) \quad (6.4)$$

and there is a similar expression for  $\partial_{S_2} F_0$  obtained by switching  $S_1$  and  $S_2$ . This agrees with the result of [10] where it was calculated using averages in the Gaussian model.

An important check is to see how the above formula reduces to the free energy (A.6) of  $S^3$  model if  $S_2 = 0$ . This means that the second set of eigenvalues ( $\mu_\alpha$ 's) disappear and we have the following relationship between the two coupling constants

$$g_s^{S^3} = \frac{g_s^{S^3/\mathbb{Z}_2}}{2}. \quad (6.5)$$

This leads to

$$\partial_S F_0^{S^3}(S) = 4\partial_{S_1} F_0^{S^3/\mathbb{Z}_2}(S_1, S_2)|_{S_1=2S, S_2=0} \quad (6.6)$$

which is indeed satisfied.

## 7 Conclusion

We have studied the matrix models that describe Chern-Simons theory on the Lens spaces  $S^3/\mathbb{Z}_p$ . We showed that the resolvent has  $p$  square root branch cuts and is thus best thought of as a  $p$ -cut single matrix model. We have found the form of the resolvent and thus the spectral curve. The spectral curve is a  $p - 1$  genus Riemann surface with 4 points deleted. We would like to lend weight to the conjecture that  $T^*(S^3/\mathbb{Z}_p)$  undergoes a large  $N$  transition to an  $A_{p-1}$  fibration over  $\mathbb{P}^1$ . We have shown that the spectral curve of the matrix model is topologically equivalent to what is expected from the mirror symmetry. However, even for  $p = 2$  we have been unable to find an explicit map between Kahler structure moduli of A-model and the periods of the B-model geometry. This must be due to the complexity of the moduli space of the manifold. It would be interesting to understand this better.

We have also calculated the free energy for the case  $p = 2$  by keeping one cut small and expanding in the appropriate small parameter. We found agreement with [10] which is a non-trivial check of our resolvent. We also showed that when one cut contains zero eigenvalues that the resolvent and free energy reduce to the case of CS theory on  $S^3$ , which is a further check of our results.

Chern-Simons theory on various manifolds can be described by a matrix model [9] and the technology introduced in this paper may find applications there. Finally, it would be intriguing if a matrix model description of the topological vertex [27] could be found and the matrix models studied in this paper may be a step in that direction.

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## A Free energy for Chern-Simons on $S^3$

In this section we will derive an expression for the leading order free energy ( $F_0$ ) in terms of the  $B$  period integral for matrix model of CS theory on  $S^3$ . There is a slight difference here compared to the matrix model with measure on the Lie algebra due to the fact that there the resolvent vanishes at infinity while for our case the resolvent is a non-zero constant at infinity. Nevertheless we still find that  $\frac{\partial F_0}{\partial S}$  is proportional to the integral over the B cycle as usual.

$F_0(S)$  is proportional to the action evaluated on-shell, if we add a single eigenvalue  $u$ , the free energy changes by

$$\Delta F_0(S) = -g_s u^2 + g_s^2 \sum_i \log \left( 2 \sinh \left( \frac{u - u_i}{2} \right) \right)^2 \tag{A.1}$$



and the corresponding change in the 't Hooft parameter is  $\Delta S = g_s$ . Therefore the derivative of the free energy with respect to  $S$  is

$$\partial_S F_0(S) = \frac{\Delta F_0}{g_s}. \tag{A.2}$$

Let's take a point at infinity  $\Lambda$ , then the following relation holds

$$\log \left( 2 \sinh \left( \frac{u - u_i}{2} \right) \right)^2 = -P \int_u^\Lambda \coth \frac{z - u_i}{2} dz - u_i, \tag{A.3}$$

where all terms except finite ones have been dropped. The last term is not present for Lie algebra matrix models, here it is due to the fact that the resolvent is finite at  $\Lambda$ . However, because (3.3) summed over  $i$  gives

$$\sum_i u_i = 0 \tag{A.4}$$

the last term in (A.3) vanishes when summed over  $i$ . Now it is easy to recognize the integral over the resolvent in (A.2)

$$\partial_S F_0(S) = \int_u^\Lambda dz (z - \omega(z)) = - \oint_B \frac{\omega(z)}{2} dz. \tag{A.5}$$

The integral can be taken explicitly with the result

$$\partial_S F_0(S) = -\frac{\pi^2}{6} + \frac{S^2}{2} + \text{Li}_2(e^{-S}), \tag{A.6}$$

where  $\text{Li}_2(x)$  is the Euler's dilogarithm function

$$\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}. \tag{A.7}$$

We will need this result when we calculate the corresponding free energy for CS theory on  $S^3/\mathbb{Z}_2$ .

## B Free energy for $S^3/\mathbb{Z}_2$ Z(2)

We now extend this analysis to Lens spaces. Let's look at how the action changes if one eigenvalue is added. Then we divide that change by  $g_s$ , which is the corresponding change in the 't Hooft parameter  $S_1$ , use (A.2) and the identity

$$\log \left( 2 \cosh \left( \frac{u - \mu_\alpha}{2} \right) \right)^2 = -P \int_u^\Lambda \tanh \frac{z - \mu_\alpha}{2} dz - \mu_\alpha, \tag{B.1}$$

$$\sum_i u_i + \sum_\alpha \mu_\alpha = 0. \tag{B.2}$$

In this way the following expression is obtained

$$\partial_{S_1} F_0(S_1, S_2) = -2 \oint_{B_1} \frac{\omega(z)}{4} dz \equiv -2\Pi_1 \tag{B.3}$$

Analogously, in order to get the derivative of the free energy with respect to  $S_2$  the integral over the  $B_2$ -cycle has to be taken. We found the resolvent to be

$$\frac{\omega(z)}{4} = \log \left( \frac{M}{2} [\sqrt{(Z+b)(Z+1/b)} - \sqrt{(Z-a)(Z-1/a)}] \right) \quad (\text{B.4})$$

with  $M = e^{-S/4}$  and

$$(Z+b)(Z+1/b) = (Z-a)(Z-1/a) + 4ZM^{-2}. \quad (\text{B.5})$$

There is one more parameter to fix, the end point  $a$  of the  $u$  cut. The contour integral over one of the  $A$ -cycles must be equal to the corresponding 't Hooft parameter. We consider the  $A_1$ -cycle

$$\oint_{A_1} \frac{\omega(z)}{4} dz = \pi i S_1. \quad (\text{B.6})$$

The cycle  $A = A_1 + A_2$  can be deformed to a contour around the logarithmic cut in the  $Z$  variable, which is proportional to the value of the other branch of the resolvent at zero,  $\tilde{\omega}(0) = -S$ . Note that the resolvent on the second sheet is the same as in (B.4) except that there is a plus sign between the two square roots. So the period integral over the  $A$ -cycle is  $\pi i S$ . Alternatively, one can take periods over  $A_1$  and  $A_2$  cycles as independent ones and obtain that  $\tilde{\omega}(0) = -S$  as a consequence. The problem of calculating integrals like (B.6) is very similar to the case of Lie algebra matrix models, in both cases they can be reduced to elliptic integrals. It is not possible to obtain the end point of the cut  $a$  as an explicit function of the 't Hooft parameters  $S_1$  and  $S_2$ , however if the size of the cuts is small then one can expand in a power series of this small parameter much like the solution of two cut Lie algebra matrix models. To this end it is better to make  $a$  and  $b$  being independent, fix the  $A_1$  and  $A_2$  periods and recover a perturbative analog of the relation (B.5).

So we introduce two small parameters  $\epsilon_1$  and  $\epsilon_2$  in the following way

$$a + \frac{1}{a} = 2(1 + \epsilon_1), \quad b + \frac{1}{b} = 2(1 + \epsilon_2). \quad (\text{B.7})$$

The resolvent up to a nonsingular term becomes

$$\frac{\omega(z)}{4} \sim \log \left( \sqrt{Z^2 + 2Z(1 + \epsilon_2) + 1} - \sqrt{Z^2 - 2Z(1 + \epsilon_1) + 1} \right). \quad (\text{B.8})$$

To take the integral (B.6) one first expands  $\omega/4$  around  $\epsilon_2 = 0$  keeping the size of the  $A_1$  cycle finite. One square root disappears so the integrals become tractable. The method is similar to one used in [3]. Note that at  $\epsilon_2 = 0$  one cut shrinks to the zero size and the resolvent matches one of Chern-Simons on  $S^3$  matrix models. We expand up to the fourth power in  $\epsilon_2$  and  $\epsilon_1$ ,

$$\begin{aligned} S_1 = & 4 \log \mu(\epsilon_1) + \frac{\epsilon_2}{2\mu(\epsilon_1)} \mathcal{A}_1 - \frac{\epsilon_2^2}{8} \left( \frac{1}{2\mu^2(\epsilon_1)} \mathcal{A}_1 + \frac{1}{\mu(\epsilon_1)} \mathcal{A}_2 \right) + \\ & + \frac{\epsilon_2^3}{24} \left( \frac{1}{2\mu^3(\epsilon_1)} \mathcal{A}_1 + \frac{1}{\mu^2(\epsilon_1)} \mathcal{A}_2 + \frac{3}{\mu(\epsilon_1)} \mathcal{A}_3 \right) - \\ & - \frac{\epsilon_2^4}{32} \left( \frac{5}{\mu(\epsilon_1)} \mathcal{A}_4 + \frac{3}{2\mu^2(\epsilon_1)} \mathcal{A}_3 + \frac{1}{2\mu^3(\epsilon_1)} \mathcal{A}_2 + \frac{3}{8\mu^4(\epsilon_1)} \mathcal{A}_1 \right) \end{aligned}$$

with  $\mu(\epsilon_1) = 1 + \epsilon_1/2$  and

$$\mathcal{A}_1 = \frac{1}{\pi i} \oint_{A_1} \frac{\sqrt{(Z-a)(Z-1/a)}}{1+Z} \frac{dZ}{Z} = 4(1 - \sqrt{\mu(\epsilon_1)}), \quad (\text{B.9})$$

$$\mathcal{A}_2 = \frac{1}{\pi i} \oint_{A_1} \frac{\sqrt{(Z-a)(Z-1/a)}}{(1+Z)^3} dZ = -\frac{\epsilon_1}{4\sqrt{\mu(\epsilon_1)}}, \quad (\text{B.10})$$

$$\mathcal{A}_3 = \frac{1}{\pi i} \oint_{A_1} \frac{Z\sqrt{(Z-a)(Z-1/a)}}{(1+Z)^5} dZ = -\frac{\epsilon_1(8+3\epsilon_1)}{128\mu^{3/2}(\epsilon_1)}, \quad (\text{B.11})$$

$$\mathcal{A}_4 = \frac{1}{\pi i} \oint_{A_1} \frac{Z^2\sqrt{(Z-a)(Z-1/a)}}{(1+Z)^7} dZ. \quad (\text{B.12})$$

Since  $A_4$  shows up in the fourth order we only need its value at  $\epsilon_1 = 0$  which is zero. The term of order  $\mathcal{O}(\epsilon_2^0)$  has been calculated using the known expression of  $S^3$  case. The next step is to expand in  $\epsilon_1$

$$\begin{aligned} S_1 = & \epsilon_1 - \frac{1}{4}\epsilon_1(\epsilon_2 + \epsilon_1) + \frac{1}{96}\epsilon_1(9\epsilon_2^2 + 15\epsilon_1\epsilon_2 + 8\epsilon_1^2) - \\ & - \frac{1}{128}\epsilon_1(5\epsilon_2^3 + 12\epsilon_2^2\epsilon_1 + 11\epsilon_2\epsilon_1^2 + 4\epsilon_1^3). \end{aligned} \quad (\text{B.13})$$

To find a similar expression for the  $A_2$  period one just has to replace  $S_1$  by  $S_2$  and switch  $\epsilon_1$  and  $\epsilon_2$  in the above formula. An important check is to recover relation (B.5) written in the form

$$S = S_1 + S_2 = 2 \log \left( 1 + \frac{\epsilon_2 + \epsilon_1}{2} \right) \quad (\text{B.14})$$

and expanded up to the fourth power in  $\epsilon_2 + \epsilon_1$ .

The two power series for the two 't Hooft parameters can be inverted giving

$$\begin{aligned} \epsilon_1 = & S_1 + \frac{1}{4}S_1(S_1 + S_2) + \frac{1}{96}S_1(3S_2^2 + 9S_1S_2 + 4S_1^2) + \\ & + \frac{1}{384}S_1(S_2^3 + 6S_2^2S_1 + 7S_2S_1^2 + 2S_1^3). \end{aligned} \quad (\text{B.15})$$

The corresponding series for  $\epsilon_2$  can be obtain from the above expression by switching  $S_1$  and  $S_2$ .

In a similar fashion one can calculate periods over the  $B$  cycles. Let's find the period  $\Pi_1$  over the  $B_1$  cycle

$$\Pi_1 = \int_{\tilde{\Lambda}}^{\Lambda} \frac{\omega(Z)}{4} \frac{dZ}{Z}, \quad (\text{B.16})$$

where  $\Lambda$  is a point at infinity on the first sheet and  $\tilde{\Lambda}$  is a point at infinity on the second sheet. Again, the first step is to expand the resolvent in power series of  $\epsilon_2$ . The integral in

the  $\mathcal{O}(\epsilon_2^0)$  term has been taken using the known result from CS theory on  $S^3$ .

$$\begin{aligned} \Pi_1 = & \frac{\pi^2}{6} - \frac{1}{2} \log^2 \mu(\epsilon_1) - \text{Li}_2 \left( \frac{1}{\mu(\epsilon_1)} \right) + \frac{\epsilon_2}{2\mu(\epsilon_1)} \mathcal{B}_1 - \\ & - \frac{\epsilon_2^2}{8} \left( \frac{1}{2\mu^2(\epsilon_1)} \mathcal{B}_1 + \frac{1}{\mu(\epsilon_1)} \mathcal{B}_2 \right) + \\ & + \frac{\epsilon_2^3}{24} \left( \frac{1}{2\mu^3(\epsilon_1)} \mathcal{B}_1 + \frac{1}{\mu^2(\epsilon_1)} \mathcal{B}_2 + \frac{3}{\mu(\epsilon_1)} \mathcal{B}_3 \right) - \\ & - \frac{\epsilon_2^4}{32} \left( \frac{5}{\mu(\epsilon_1)} \mathcal{B}_4 + \frac{3}{2\mu^2(\epsilon_1)} \mathcal{B}_3 + \frac{1}{2\mu^3(\epsilon_1)} \mathcal{B}_2 + \frac{3}{8\mu^4(\epsilon_1)} \mathcal{B}_1 \right), \end{aligned} \quad (\text{B.17})$$

where

$$\mathcal{B}_1 = \int_{\tilde{\Lambda}}^{\Lambda} \frac{\sqrt{(Z-a)(Z-1/a)} dZ}{1+Z} \frac{1}{Z}, \quad (\text{B.18})$$

$$\mathcal{B}_2 = \int_{\tilde{\Lambda}}^{\Lambda} \frac{\sqrt{(Z-a)(Z-1/a)} dZ}{(1+Z)^3}, \quad (\text{B.19})$$

$$\mathcal{B}_3 = \int_{\tilde{\Lambda}}^{\Lambda} \frac{Z \sqrt{(Z-a)(Z-1/a)} dZ}{(1+Z)^5}, \quad (\text{B.20})$$

$$\mathcal{B}_4 = \int_{\tilde{\Lambda}}^{\Lambda} \frac{Z^2 \sqrt{(Z-a)(Z-1/a)} dZ}{(1+Z)^7}. \quad (\text{B.21})$$

Then one sends  $\Lambda$  to infinity and takes the finite part, which is then expanded in powers of  $\epsilon_1$

$$\begin{aligned} \mathcal{B}_1 = & -\log 16 + \frac{1}{2}(-1 + \log(\epsilon_1/8))\epsilon_1 + \frac{1}{32}(-1 - 2\log(\epsilon_1/8))\epsilon_1^2 + \\ & + \frac{1}{384}(5 + 6\log(\epsilon_1/8))\epsilon_1^3 + \mathcal{O}(\epsilon_1^4), \end{aligned} \quad (\text{B.22})$$

$$\mathcal{B}_2 = \frac{1}{2} + \frac{1}{8}\epsilon_1 \log(\epsilon_1/8) - \frac{1}{32}(1 + \log(\epsilon_1/8))\epsilon_1^2 + \mathcal{O}(\epsilon_1^3), \quad (\text{B.23})$$

$$\mathcal{B}_3 = \frac{1}{16} + \frac{1}{64}(1 - 2\log(\epsilon_1/8))\epsilon_1 + \mathcal{O}(\epsilon_1^2), \quad (\text{B.24})$$

$$\mathcal{B}_4 = \frac{1}{96} + \mathcal{O}(\epsilon_1). \quad (\text{B.25})$$

Combining this all together and plugging in the expressions for  $\epsilon_1$  and  $\epsilon_2$  as functions of  $S_1$  and  $S_2$  and using (B.3) we get

$$\partial_{S_1} F_0(S_1, S_2) = -S_1(1 + \log 2) + 2S_2 \log 2 + S_1 \log S_1 + \frac{1}{8}(S_1 + S_2)^2 + \quad (\text{B.26})$$

$$+ \frac{1}{576}(3S_2^3 + 18S_2^2 S_1 + 9S_2 S_1^2 + 2S_1^3) + \mathcal{O}(S^5), \quad (\text{B.27})$$

which is in agreement with [10]. Note there is no terms of order  $\mathcal{O}(S^4)$ .

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